ITERATION METHODS OF SOLVING SOME PROBLEMS IN THERMOSTATICS

A. G. Ramm

UDC 534.241

We shall consider the following thermostatic problems:

$$\Delta u = 0 \text{ in } D_i,\ u|_S = f; \tag{1}$$

$$\Delta u = 0 \text{ in } D_e,\ u|_S = f. \tag{2}$$

where D_i is an arbitrary region bounded by a piecewise closed surface S; $u(x)$ is the temperature field; $x = (x_1, x_2, x_3)$; and D_e is a region external to the surface S. When the boundary condition is of the form $\partial u/\partial n|_S = f$, the problems defined by Eqs. (1) and (2) will be denoted by Eqs. (1a) and (2a). We propose an iteration processes for the solutions of the problems defined by Eqs. (1), (2), (1a), and (2a), and derive approximate formulas for the thermal resistance of a body of arbitrary shape. It is shown that the solution of Eq. (1) can be written in the form

$$u(x) = \int_S \mu(t) \frac{\partial}{\partial n_t} \cdot \frac{1}{4\pi r_{xt}}\, dt,\ \mu(t) = \lim_{n\to\infty} \mu_n(t), \tag{3}$$

where $r_{xt} = |x-t|$, n_t is the external normal to the surface passing through the point t; and $\mu_n(t)$ can be calculated by the following iteration process:

$$\mu_{n+1} = B\mu_n - \frac{2f}{1+a};\ \mu_0 = -\frac{2f}{1+a}, \tag{4}$$

$$B = \frac{A + aI}{1+a};\ A\mu = \int_S \mu(t) \frac{\partial}{\partial n_t} \cdot \frac{1}{2\pi r_{st}}\, dt, \tag{5}$$

where $a > 0$ is an arbitrary number which can be chosen so that the process defined by Eq. (4) converges as fast as possible. For example, $a = 4/3$ when S is a sphere.

The solution of the problem defined by Eq. (1a) subjected to the condition $\int_S f(s)ds = 0$ (this condition must be satisfied if the problem is to have a solution) can be obtained from the formula:

$$u(x) = \int_S \frac{\sigma(t)\, dt}{4\pi r_{xt}},\ \sigma(t) = \lim_{n\to\infty} \sigma_n(t), \tag{6}$$

where $\sigma_n(t)$ is found from

$$\sigma_{n+1} = -\int_S \frac{\partial}{\partial n_s} \cdot \frac{1}{2\pi r_{st}} \sigma_n(t)\, dt + 2f,\ \sigma_0 = 2f. \tag{7}$$

For the electrostatic capacitance of a body (which differs from the thermal conductance of this body only by a numerical factor) we have obtained the following approximate formula:

$$C = 4\pi |S|^2 \left\{ \int_S \int_S \frac{dtds}{r_{ts}} \right\}^{-1},\ |S| = \text{mes}\, S. \tag{8}$$

The solution of Eq. (2) can be obtained from the formula:

$$u(x) = \int_S \mu(t) \frac{\partial}{\partial n_t} \cdot \frac{1}{4\pi r_{xt}}\, dt + \frac{\alpha}{|x|},\ \mu = \lim_{n\to\infty} \mu_n, \tag{9}$$

Institute of Rigorous Mechanics and Optics, Leningrad. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 20, No. 5, pp. 936-937, May, 1971. Original article submitted September 23, 1969; revision submitted June 16, 1970.

where

$$\mu_{n+1} = -A\mu_n + \int_S \mu_n(t)\,dt + 2\left(f - \frac{\alpha}{|s|}\right),\quad \mu_0 = 2\left(f - \frac{\alpha}{|s|}\right), \tag{10}$$

and the constant α is obtained from the condition $\int_\Gamma \mu(t)dt = 0$.

The solution of the problem defined by Eq. (2a) is found from Eq. (6), in which $\sigma(t)$ is calculated from Eqs. (4)-(5) with A replaced by the operator A*:

$$A^*\sigma = \int_S \frac{\partial}{\partial n_s} \cdot \frac{1}{2\pi r_{st}}\,\sigma(t)\,dt. \tag{11}$$

All the above iteration processes converge at the rate of a geometric progression.

The approximate formula for the thermal resistance between two surfaces, S_1 and S_0 (S_0 surrounds S_1) is

$$\frac{1}{C} = \frac{1}{|S_1|}\int_{S_1} u(s)\,ds - \frac{1}{|S_0|}\int_{S_0} u(s)\,ds, \tag{12}$$

where

$$u(x) = \int_{S_0+S_1} \frac{\sigma_0(t) + A^*\sigma_0(t)}{4\pi r_{xt}}\,dt,\quad \sigma_0(t) = \begin{cases} \dfrac{1}{2|S_1|}, & t \subset S_1, \\ -\dfrac{1}{2|S_0|}, & t \subset S_0, \end{cases} \tag{13}$$

and $|S|$ represents the area of the surface S. The formula given by Eq. (8) was used on the Minsk-22 computer to calculate the capacitance of a unit cube. The result was $C_1 = 0.64$ which differs from the published result $C_1 = 0.65$ by less than 3%. We have also calculated the capacitance of a circular cylinder of length 2L and radius a for $L/a \geq 0.1$. The error in the zero-order approximation did not exceed 0.03; $C_1 = C/4\pi$.

The foregoing leads us to conclude that the proposed iteration processes are very effective for the numerical and approximate analytic solution of problems in thermostatics.